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A Path-Dependent Approach to Colored Magnetic Monopoles

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Abstract

We extend Abelian gauge theory of magnetic monopoles to the non-Abelian case, by means of Mandelstam's path-dependent approach to Yang–Mills fields. In this framework, the $SU(3)$ generalization of Dirac's relationship plays the role of a consistency condition for the theory.

It has been realized lately that gauge theories with magnetic monopoles seem to play a relevant role in the picture of hadrons as one-dimensional objects (strings) (Nielsen and Olesen, 1973; Nambu, 1969; Parisi, 1975). Moreover, such a model can give a very simple and clear explanation of quark confinement (Parisi, 1975; Nambu, 1974). Because of the importance of $SU(3)$ symmetry in describing hadron properties, one feels that an extension of the string model to the case on non-Abelian gauge theories should be of more physical interest.

In such a line of thought, Dirac's theory of magnetic monopoles was recently and independently extended by Venturi (1975) and Eguchi (1975) to the case of non-Abelian color gauge symmetry. In the present paper, we want to deal with unbroken $SU(3)$ gauge field theory on the basis of Bialynicki-Birula's (1962) and Mandelstam's (1962) formulation of Yang-Mills fields. Therefore, our approach to colored monopoles is a generalization of Cabibbo and Ferrari's (1962) Abelian theory of magnetic poles.

Let us consider a system of monopoles (quarks), carrying $SU(3)$ (color) quantum numbers, interacting through charged gluons, and assume that no electric current is present. In the path-dependent formalism, the equations of motion for this system are given by (Bialynicki-Birula, 1962; Mandelstam, 1968)

$$\partial^{\mu} F^{a}_{\mu\nu}(x, P) = 0 \tag{1}$$

$$\partial^{\mu} \tilde{F}^{a}_{\mu\nu}(x, P) = g_{\nu}{}^{a} \tag{2}$$

The tensor $F^a_{\mu\nu}(x, P)$ is the Yang–Mills field associated to the gluons and $g_\nu{}^a$ is the (colored) magnetic current.[1] We denote by P a path extending from point x to $-\infty$. The dual tensor is defined as usual::

$$\tilde{F}^a_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma} \tag{3}$$

The main difference between equations (1) and (2) and the corresponding ones for electromagnetism is that, in this case, the Yang–Mills field, being a charged field, is path dependent, too. (Bialyncki-Birula, 1962; Mandelstam, 1968).

In the absence of monopoles [i.e., when the right-hand side of equation (2) is zero], the connection between the path-dependent (and gauge-invariant) tensor $F^a_{\mu\nu}(x, P)$ and the path-independent one, $f^a_{\mu\nu}(x)$, is given by the matrix relation

$$F_{\mu\nu}(x, P) = U^{-1}(x, P)f_{\mu\nu}(x)U(x, P) \tag{4}$$

where $U(x, P)$ is the unitary matrix [in $SU(3)$ space]:

$$U(x, P) = T\exp\left(ie\int_P^x d\xi^\rho A_\rho{}^a\lambda_a\right) \tag{5}$$

Besides, we introduced the notation $F_{\mu\nu} = F^a_{\mu\nu}\lambda_a$. The matrices λ_a are the usual Gell-Mann matrices and T is a symbol that prescribes ordering for the λ_a's from the beginning to the end of the path when one expands the exponential. The $A_\mu{}^a$ are the $SU(3)$ components of the (path-independent) four-potential entering the relation for $f^a_{\mu\nu}$:

$$f^a_{\mu\nu} = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + e\epsilon_{abc}A_\mu{}^b A_\nu{}^c \tag{6}$$

Here, ϵ_{abc} are $SU(3)$ structure constants.

In general, a colored field $\psi(x, P)$ exhibits the following path dependence (Bialynicki-Birula, 1962):

$$\delta_z\psi(x, P) = -ie\,[F_{\mu\nu}(z, P'),\ \psi(x, P)]\,\delta\sigma^{\mu\nu} \tag{7}$$

where $\delta_z\psi(x, P)$ is the change of $\psi(x, P)$ when P is deformed by an infinitesimal area $\delta\sigma^{\mu\nu}$ in the neighborhood of point $z \in P$, and P' coincides with P but ends at z. For a finite change in the path, we obtain the integral relation

$$\psi(x, P'') = V(P'', P)\psi(x, P)V^{-1}(P'', P) \tag{8}$$

where P'' is the new path obtained through (finite) deformation of P and the operator $V(P'', P)$ is defined by

$$V(P'', P) = \exp\left[-ie\int_S d\sigma^{\mu\nu}F_{\mu\nu}(z, P')\right] \tag{9}$$

[1] As usual, Greek letters denote space-time indices (running from 0 to 3), whereas Latin letters correspond to $SU(3)$ (color) indices ($a, b, \ldots = 1, \ldots$). Summation over repeated indices (both Greek and Latin) is always understood.

In equation (9), the integral must be evaluated on an *arbitrary* open surface delimited by the closed path P''-P. Obviously, equation (8) makes sense only if the integral in equation (9) is surface independent. It is easy to realize that such a requirement is satisfied if

$$\exp\left[-ie\int_{S'} d\sigma^{\mu\nu}F_{\mu\nu}(x,P')\right]=I \tag{10}$$

where I is the unity matrix, and S' is the closed surface $S_1 - S_2$ (with S_1, S_2 any two of the infinite surfaces having the closed path $P'' - P$ as contour). By applying Green's theorem, we may change the surface integral into an integral over the volume V enclosed by S'. Therefore, condition (10) is equivalent to

$$e\int_V dV\,\partial^\mu \tilde{F}^a_{\mu\nu}(z,P') = 2\pi n \tag{11}$$

It follows that, in the path-dependent theory of Yang–Mills field without monopoles, equation (10) is automatically satisfied.

Now let us assume, in analogy with the Abelian case in the path-dependent approach (Cabibbo and Ferrari, 1962), that the path dependence of $\psi(x,P)$ is still given by equations (8) and (9) even when monopoles are put in the theory, i.e., when $g_\nu{}^a \neq 0$ in equation (2).[2] For consistency, we must *impose* condition (11), which now becomes

$$e\int_V dV\,g_\nu{}^a\lambda_a = 2\pi n \tag{12}$$

Therefore, in the $SU(3)$ case, in order to ensure surface independence of equation (8), one needs the following relation between electric and magnetic charges (Venturi, 1975; Eguchi, 1975):

$$eg = 3\pi n \tag{13}$$

Equation (13) is a generalization of Dirac's relationship $eg = 2\pi n$. It was previously derived (Venturi, 1975; Eguchi, 1975) by imposing single valuedness of the wave function, whereas in the present framework – as in the Abelian Cabibbo–Ferrari theory – it plays the role of a consistency condition.

Once equation (11) is satisfied, all the machinery of the path-dependent treatment of Yang–Mills field (Bialynicki-Birula, 1962; Mandelstam, 1968) – based essentially on equations (8) and (9) – can be straightforwardly extended, without essential modifications, to the present case of interaction with magnetic monopoles, and we do not dwell on this.

In the last months, compelling experimental evidence (Blietschau et al., 1976) has been given of the existence of charmed particles. Thence, it seems that $SU(4)$ is more appropriate in describing symmetry properties of hadrons.

[2] However, notice that – contrary to the Abelian case (Cabibbo and Ferrari, 1962) – now it seems impossible, in general, to express explicitly the tensor $F_{\mu\nu}$ in terms of potentials.

It may be useful, therefore, to introduce, by an analogous procedure, "charmed" magnetic monopoles interacting with the Yang-Mills field. This goal can be readily achieved with obvious changes in the formalism: For example, one has to replace equation (13) by $eg = 4\pi n$ for monopoles endowed with $SU(4)$ quantum numbers.

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